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PLITHOGENIC FUZZY GRAPH: A STUDY OF FUNDAMENTAL PROPERTIES AND POTENTIAL APPLICATIONS

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Abstract. A fuzzy graph is a symmetric binary fuzzy relation on a fuzzy subset. It has been successfully involved in a number of applications in data science, social networks, neural systems, and many more. There exist some uncertainties in fuzzy graphs where intuitionistic and picture fuzzy graphs are introduced to overcome these uncertainties. However, there are still some other uncertainties in real-world problems unresolved by fuzzy, intuitionistic, and picture fuzzy graphs. To overcome these uncertainties, a Plithogenic fuzzy graph (PFG) is introduced in this work, as the most generalized form of fuzzy graph. First, we present some basic definitions such as order, size, degree, path, cycle, complete, and complete bipartite graph and generalize the first Zagreb index in the environment of PFG. Moreover, some applications and comparative studies are investigated for the generalized first Zagreb index in the environment of PFG. PFG environment of aggregation did not entertain the information of any problem having more than one expert for the aggregation of attributes. It contributes to the basic comprehension of this innovative idea by introducing and establishing the fundamental features of PFGs. That is why we introduce the Plithogenic environment of PFG which accomplishes a number of noteworthy firsts in the study of PFGs. By integrating Plithogenic sets, this work goes beyond traditional fuzzy and intuitionistic fuzzy graphs, opening the door to a new paradigm in graph theory.

1. **Introduction.**

The fuzzy set was first introduced by Zadeh [51], describing the elements of a set with the help of a positive membership value from the interval $[0, 1]$. Operations of crisp sets such as union, intersection, convexity, relations, and many more were also extended to fuzzy sets [52]. Fuzzy logic is a generalization of a crisp set or boolean logic. An intuitionistic fuzzy set (IFS) introduced by Atanassov is a generalization of fuzzy sets [6]. An intuitionistic deals with membership and non-membership value but there are some real-world problems in which IFSs fail due to their uncertain nature. To deal with hesitancy and vagueness in informative

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data a new concept Picture fuzzy set consisting of membership, abstinence, and non-membership degree was introduced by Coung [11]. In some cases, the sum of all these degrees extended from 1 to handle such vagueness a new concept called Picture fuzzy set was established [47]. Smarandache [42] presented a new set known as the Pithogenic set whose elements are characterized by many attributes. Plithogenic is a generalization of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets. Plithogenic aggregation operators are a type of fuzzy operator that is constructed using a linear combination of T-norm and T-conorm operators. The aggregation operators of Plithogenic sets are related to the contradiction degree, which is used to maximize the accuracy level. For more generic fuzzy environments, one can see Meng et al. [28], Dong et al. [13], and Murugan et al. [30].

Graph theory is a very important research area due to its wide applications in many fields, such as computer science, mathematics, and engineering. To handle the uncertainty fuzzy graph was introduced. Later on, to handle uncertainty in different real-world problems, intuitionistic, and picture fuzzy graphs were introduced.

Fuzzy sets have many applications in different fields of life such as system analysis [31], safety and reliability engineering [20], man-machine interaction [24], decision-making [34, 33, 9], gene subset selection [25], and many more. A fuzzy graph was introduced by Rosenfeld [38] for the first time. He defined its fundamental concepts and properties like trees and bridges. Bhutani and Rosenfeld [8] and Mathew et al. [27] addressed the applications of geodesics and bipolar fuzzy graphs in many networks. Akram and Khan [4] established a new concept of fuzzy graph known as complex Dombi fuzzy graph. On the other hand, fuzzy graphs have applications in decision-making, human trafficking [40, 29], the social network era, cybernetics [26], transport planning [39], and many more. In IFS a new degree of negative or non-membership was introduced, this concept has got special attention among the researchers due to its negative membership [44, 23, 14]. Some basic operations of IFS like union, intersection, and many others are also discussed [12]. The concept of IFS was extended in the form of a Pythagorean fuzzy set by Yager [50] and Zhang and Xu [53].

A graph is classified into many other types such as path [10], cycle [35], complete graph [15], and so on. Some basic operations of graphs like union, intersection, sum, vertex, edge deletion, indices like Zagreb index [5] and many more are discussed. Graph theory is an attractive approach for researchers because of its real-life applications [46]. Parvathi and Karunambigai [36] defined the intuitionistic fuzzy graph precisely. Parvathi, Karunambigai, and Attanossov also introduced some graph operations like complement and some other properties of a graph [2, 37]. Applications of intuitionistic fuzzy graphs are in many areas such as decision support systems, communication networks, etc [3]. Akram and Alshehri [1] redefined intuitionistic fuzzy cycles, intuitionistic fuzzy trees, and many corresponding properties. The generalized version of an intuitionistic fuzzy graph is a picture fuzzy graph that discusses all three membership, abstinence, and non-membership degrees [54]. Recently, many authors addressed different operations of graphs such as graph symmetries [41, 18, 54] as well as applications of picture fuzzy graphs [49]. The PFG has also expanded for topological indices, which is a numerical quantity that describes the structural properties of a graph based on molecules. Topological indices have many properties one of them is all isomorphic graphs have the same values [7]. Wiener [48] described the concept of topological indices known as the first Zagreb index.

In this study, we introduce a Plithogenic fuzzy graph (PFG) along with the order, size, and degree of a PFG. Furthermore, we develop path, cycle, complete, and complete bipartite graphs in the environment. In this regard, defining the generalized first Zagreb index also contributes to the application of a network of patients affected by lung pleural effusion.

1.1. **Motivations.** Kauffman [21] delineated the idea of fuzzy graph theory, which consists of membership values that have a number of applications in different fields like data science, communication, clustering, etc. Due to its single-value nature, there exist some uncertainties to overcome these uncertainties. Parvathi et al. [37] presented an intuitionistic fuzzy graph that has many applications in different areas like decision support systems, communication networks, etc. To overcome the uncertainties in the intuitionistic fuzzy graph picture fuzzy graph was offered by Zuo et al. [54]. Picture fuzzy graph has three aggregated values known as membership, abstinence, and non-membership value but all three values depend on each other because of its restriction $0 \leq \{\text{membership} + \text{abstinence} + \text{non-membership}\} \leq 1$. Moreover, all of these three environment values are aggregated by one expert. This environment of aggregation did not entertain the information of any problem having more than one expert for the aggregation of attributes.

In order to resolve such uncertainties, we introduce a PFG, in which there are three or more three experts who aggregate the attributes. All these experts are independent so, because of their independent nature and more number of experts aggregating the attributes we get more generalized results as compared to other environments.

The fuzzy graph has very useful applications in data mining, image segmentation, clustering, image capturing, networking, communication, planning, and scheduling. The intuitionistic fuzzy graph is reliable due to its non-membership value with a wide range of applications in pattern recognition, machine learning, decision-making, and market prediction. Moreover, the picture fuzzy graph is comparatively more reliable due to its abstinence value which helps to measure required attributes with a better accuracy. It is widely applicable in networking and enables the solving of three-dimensional problems.

In this research, we let a scenario in which a network of patients affected by any particular disease and every patient is aggregated by three doctors such that the doctors aggregate the symptoms of the disease and then compare them with others and with the help of the first general Zagreb index in the environment of PFG we can calculate common symptoms of that decease can shown in any other newly affected patient.

1.2. **Contributions.** The purpose of this study is to introduce a PFG, as the Plithogenic set is the most generalized form of fuzzy set and the graph created by the relation of a most generalized set is also a generalized graph that helps us to solve a real-world problem with less of information as compared to other environments. The approach of aggregating any attribute represents the most appropriate aggregation among the different choices of aggregation in existing environments. Consequently, the main object of this research is to present a PFG. Ultimately, the following are the basic achievements of this work:

1. Establishing a PFG,
2. Developing path, cycle, complete and complete bipartite graph in the framework of PFG,
3. Presenting degree, order, and size of a PFG,
4. Developing the first general Zagreb index for a PFG,
5. Establishing the first general Zagreb index in the framework of PFG for path, cycle, and complete graph,
6. Discussing an application of a network of patients affected by lung pleural effusion to check the reliability of our proposed work,
7. Comparing our proposed methodology with already existing methods to demonstrate its efficiency.

1.3. **Structure of the manuscript.** The structure of this paper is set up as follows. In Section 2, basic definitions and preliminaries of PFG like Plithogenic path graph, Plithogenic cycle graph, Plithogenic complete graph, and Plithogenic complete bipartite graph. Furthermore, some basic terms of graphs for PFG are discussed, including vertex set, edge set, degree set, order, size, and degree of vertex of a PFG. Section 3 describes the main results in which the first general Zagreb index is presented in the framework of PFG, and also discusses this index for the families of graphs like path, cycle, and complete graph. In Section 4, we discuss the application of a network of patients affected by lung pleural effusion in order to prove our proposed work and also show that if the number of edges or the number of vertices of PFG increases then the first general Zagreb index will increase too. Additionally, we compare our proposed work with already existing methods and examine how our proposed work is better than already existing methods. Section 5 summarizes our proposed work and its application, and finally, Section 6 gives a direction for future work and discusses some open problems for the PFG environment.

1.4. **Problem statement.** Here, the Plithogenic fuzzy graph is delineated because of its greater variety of addressing problems compared to all other (already discussed) fuzzy systems such as fuzzy graphs, intuitionistic fuzzy graphs, and picture fuzzy graphs. Plithogenic fuzzy graph is more helpful in decision-making due to its generalized nature. On the other hand, the other fuzzy values are aggregated by one expert. This environment of aggregation does not entertain the information of any problem having more than one expert for the aggregation of attributes.
To resolve such uncertainties, PFG is introduced, in which there are three or more three experts who aggregate the attributes. All these experts are independent due to their independent nature and with more experts aggregating the attributes we get more generalized results as compared to other environments.

2. **Basic Definitions.**

Definition 2.1. A PFG $G(V, \mu, \rho)$ is a non empty set V together with a pair of functions $\mu_1, \mu_2, \mu_3 : V \rightarrow [0,1]$ and $\rho_1, \rho_2, \rho_3 : V \times V \rightarrow [0,1]$ such that $\forall x, y$ in V, $(\rho_1(x_1, y_1)\,(\rho_2(x_2, y_2))\,(\rho_3(x_3, y_3))) \leq (\min\{\mu_1(x_1), \mu_1(y_1)\}\,(\min\{\mu_2(x_2), \mu_2(y_2)\}\,(\min\{\mu_3(x_3), \mu_3(y_3)\})))$. Vertex set and edge set is denoted by $V(G)$and $E(G)$, respectively. An example of PFG is represented in Figure 1.

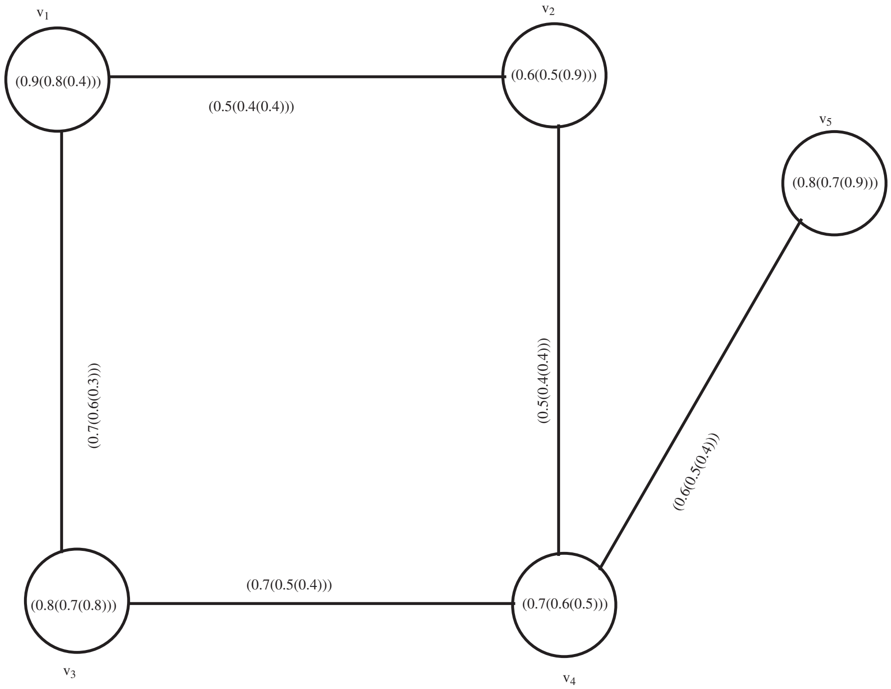

Figure 1. Plithogenic fuzzy graph.

Definition 2.2. The order of a PFG is denoted and defined as $O(PFG) = \sum_{i=1}^{n}(x_i(y_i(z_i)))$. The order of Figure 1 is $(0.9 + 0.6 + 0.8 + 0.7 + 0.8(0.8 + 0.5 + 0.7 + 0.6 + 0.7(0.4 + 0.9 + 0.8 + 0.5 + 0.9))) = (3.8(3.3(3.5)))$.

Definition 2.3. The size of a PFG is denoted and defined as $S(PFG) = \sum_{i,j=1}^{n}(x_{ij}(y_{ij}(z_{ij})))$. The size of Figure 1 is $(0.5 + 0.7 + 0.5 + 0.7 + 0.6(0.4 + 0.6 + 0.4 + 0.5 + 0.5(0.4 + 0.3 + 0.4 + 0.4 + 0.4))) = (3.0(2.4(1.9)))$.

Definition 2.4. The degree of the vertex of PFG is equal to the sum of values of all incident edges of that vertex. Denoted as $d(x) = \sum_{xy\in E(G)}(\rho_1(xy)(\rho_2(xy)(\rho_3(xy))))$. According to Figure 1,

$$\begin{aligned}
d(v_1) &= (0.5(0.4(0.4))) + (0.7(0.6(0.3))) = (1.2(1.0(0.7))),\\
d(v_2) &= (0.5(0.4(0.4))) + (0.5(0.4(0.4))) = (1.0(0.8(0.8))),\\
d(v_3) &= (0.7(0.6(0.3))) + (0.7(0.5(0.4))) = (1.4(1.1(0.7))),\\
d(v_4) &= (0.7(0.5(0.4))) + (0.5(0.4(0.4))) + (0.6(0.5(0.4))) = (1.8(1.4(1.2))),\\
d(v_5) &= (0.6(0.5(0.4))).
\end{aligned} \tag{1}$$

Lemma 2.5. *Let PFG be a graph with n vertices, where $d(v_i)$ is degree and $S(PFG)$ is size of graph, and under Plithogenic environment, we have*

$$\sum_{i=1}^{n} d(v_i) = 2(S(PFG)).$$

For example, in Figure 1,

$$\begin{aligned}
&\sum_{i=1}^{n} d(v_i) = (6.0(4.8(3.8))),\\
&S(PFG) = (3.0(2.4(1.9))),\\
&2(S(PFG)) = (6.0(4.8(3.8))), \qquad (2)\\
&\sum_{i=1}^{n} d(v_i) = 2(S(PFG)).
\end{aligned}$$

Definition 2.6. A PFG is a path graph whose vertices can be listed in the order $(x_1(y_1(z_1))), (x_2(y_2(z_2))), (x_3(y_3(z_3))), \ldots, (x_n(y_n(z_n)))$ such that the edges of the graph are $((x_1(y_1(z_1))), (x_{i+1}(y_{i+1}(z_{i+1}))))$ where $i = 1, 2, 3, \ldots, i-1$. In other words, a path P of length n is a sequence of distinct nodes $(x_1(y_1(z_1))), (x_2(y_2(z_2))), (x_3(y_3(z_3))), \ldots, (x_n(y_n(z_n)))$ such that $\mu\left((x_1(y_1(z_1))), (x_{i+1}(y_{i+1}(z_{i+1})))\right) > 0$ where $i = 1, 2, 3, \ldots, i-1$. The Plithogenic path graph is represented in Figure 2.

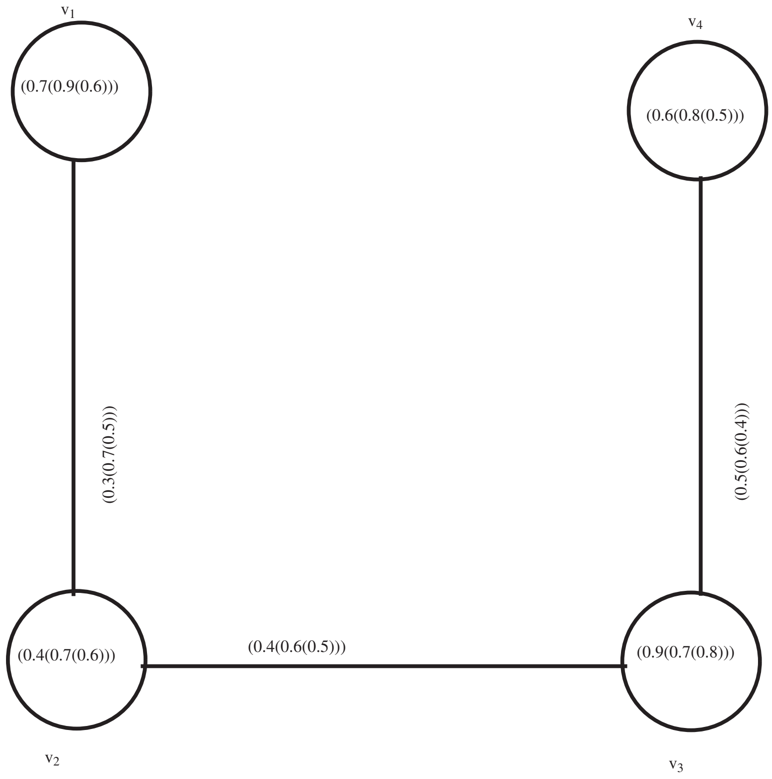

FIGURE 2. Plithogenic path graph.

Lemma 2.7. *Let P_n be a path graph with $n \geq 2$. Then, the degree of its vertices under Plithogenic environment is:*

$$d(v_i) = d(v_1) \text{ if } i = 1,$$

$$\begin{aligned} d(v_i) \leq{} & (\min\{\mu_1(x_i), \mu_1(y_i)\} \\ & + \min\{\mu_1(x_{i+1}), \mu_1(y_{i+1})\} (\min\{\mu_2(x_i), \mu_2(y_i)\} \\ & + \min\{\mu_2(x_{i+1}), \mu_2(y_{i+1})\} (\min\{\mu_3(x_i), \mu_3(y_i)\} \\ & + \min\{\mu_3(x_{i+1}), \mu_3(y_{i+1})\}))) \text{ if } \ i = 2, 3, 4, \ldots, n-1, \\ d(v_i) ={} & d(v_n) \text{ if } \ i = n. \end{aligned}$$

Definition 2.8. A PFG is a cycle graph whose vertices can be listed in the order $(x_1(y_1(z_1))), (x_2(y_2(z_2))), (x_3(y_3(z_3))), \ldots, (x_n(y_n(z_n)))$ such that the edges of the graph are $((x_1(y_1(z_1))), (x_{i+1}(y_{i+1}(z_{i+1}))))$ where $i = 1, 2, 3, \ldots, i$ and $i + 1 = i$. The Plithogenic cycle graph is displayed in Figure 3.

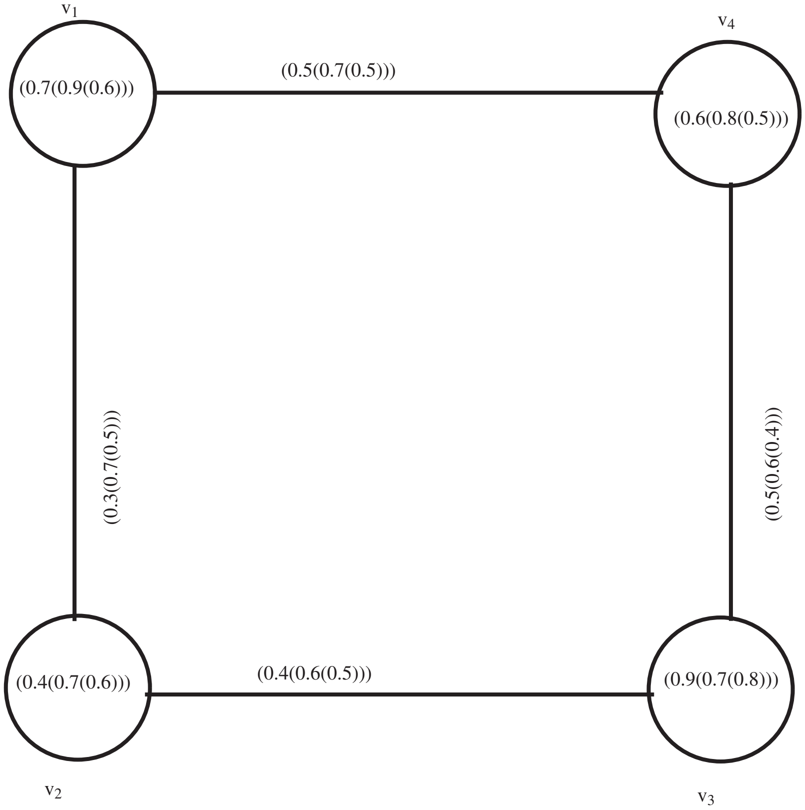

FIGURE 3. Plithogenic cycle graph.

Lemma 2.9. *Let C_n be a cycle graph with $n \geq 2$. Then, the degree of its vertices under Plithogenic environment is:*

$$\begin{aligned} d(v_i) \leq{} & (\min\{\mu_1(x_i), \mu_1(y_i)\} \\ & + \min\{\mu_1(x_{i+1}), \mu_1(y_{i+1})\} (\min\{\mu_2(x_i), \mu_2(y_i)\} \\ & + \min\{\mu_2(x_{i+1}), \mu_2(y_{i+1})\} (\min\{\mu_3(x_i), \mu_3(y_i)\} \\ & + \min\{\mu_3(x_{i+1}), \mu_3(y_{i+1})\}))) \text{ if } \ i = 1, 2, 3, 4, \ldots, n. \end{aligned}$$

Definition 2.10. A complete graph is said to be a complete PFG if

$$\begin{aligned}&(\rho_1(x_1, y_1)\,(\rho_2(x_2, y_2))\,(\rho_3(x_3, y_3)))\\&= (\min\{\mu_1(x_1), \mu_1(y_1)\}\,(\min\{\mu_2(x_2), \mu_2(y_2)\}\,(\min\{\mu_3(x_3), \mu_3(y_3)\}))) \,.\end{aligned}$$

The Plithogenic complete graph is shown by Figure 4.

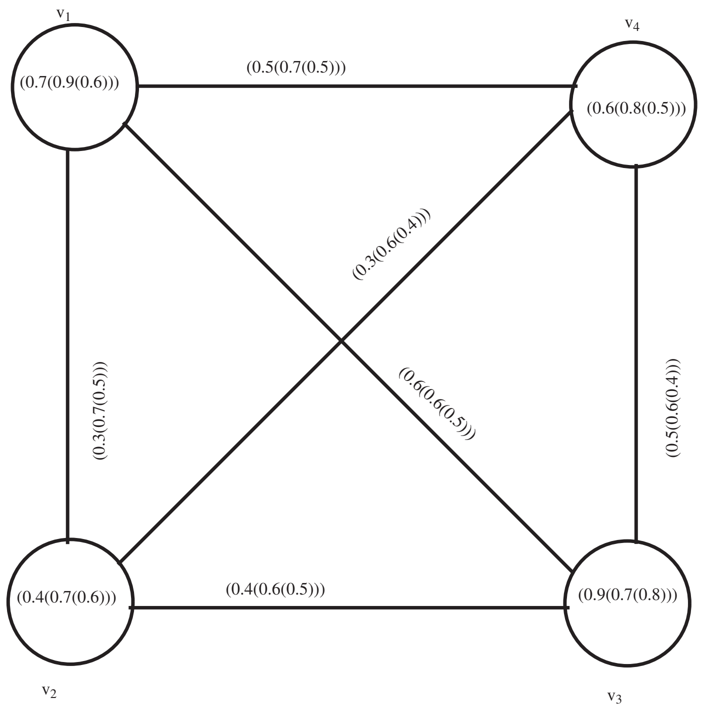

FIGURE 4. Plithogenic complete graph.

Lemma 2.11. *Let K_n be a complete graph with $n \geq 2$. Then, the degree of its vertices under Plithogenic environment is:*

$$d(x) = \sum_{xy \in E(PFG)} (\rho_1(xy)(\rho_2(xy)(\rho_3(xy)))), \quad \text{if} \;\; i = 1, 2, 3, 4, \ldots, n. \tag{3}$$

Definition 2.12. A complete bipartite graph is a graph that can be divided into two parts, V and U, such that no two vertices in the same part are connected by an edge, and every possible edge between a vertex in V and a vertex in U is present in the graph.

Definition 2.13. A complete bipartite graph is said to be a complete bipartite PFG if

$$
\begin{aligned}
&(\rho_1(x_1,y_2)(\rho_2(x_2,y_2)\rho_3(x_3,y_3))) \\
&= \left(\min\{\mu_1(x_1),\mu_1(y_1)\}\left(\min\{\mu_2(x_2),\mu_2(y_2)\}\left(\min\{\mu_3(x_3),\mu_3(y_3)\}\right)\right)\right).
\end{aligned}
$$

The Plithogenic complete bipartite graph is depicted in Figure 5.

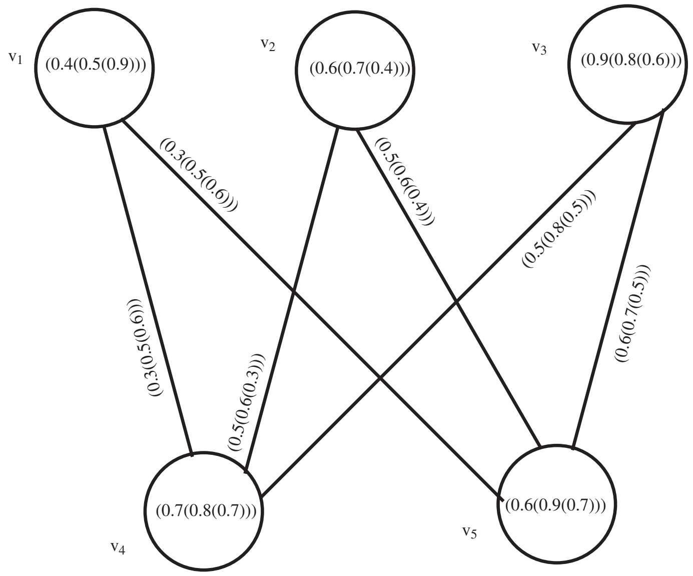

FIGURE 5. Plithogenic complete bipartite graph.

Lemma 2.14. *Let $K_{m,n}$ be a complete bipartite graph with $n \geq 2$. Then, the degree of its vertices under Plithogenic environment is: $d(v_i) = \sum_{i=1}^{m}\sum_{j=1}^{n} v_i u_j$ for vertex set V and $d(u_j) = \sum_{j=1}^{n}\sum_{i=1}^{m} u_j u_i$ for vertex set U.*

3. Main results.

Definition 3.1. Li and Zhang [22] introduced the concept of the first general Zagreb index, denoted and defined as:

$$M_\alpha(G) = \sum_{i=1}^{n} d^\alpha(v_i). \tag{4}$$

We introduced the concept of the first general Zagreb index for the environment of PFG.

Definition 3.2. The first general Zagreb index is denoted as

$$M_\alpha(PFG) = \sum_{i=1}^{n}(P(x_i(y_i(z_i))))d^\alpha(x_i(y_i(z_i))). \tag{5}$$

In the following theorem, we discuss the first general Zagreb index of the Plithogenic path graph.

Theorem 3.3. *Let P_n be a Plithogenic path graph with $V = \{\mu_1, \mu_2, \mu_3, \ldots, \mu_n\}$ the set of vertices and $E = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_n\}$ be the set of $n-1$ edges, such that $\mu_1 = (x_1(y_1(z)_1))$ and $\rho_1 = (x_1(y_1(z)_1))(x_2(y_2(z)_2))$. Then the Plithogenic version of the first general Zagreb index is:*

$$M_\alpha(PFG) = (P(\mu_1))d^\alpha(\mu_1) + \sum_{i=2}^{n-1}(P(\mu_i))d^\alpha(\mu_i) + (P(\mu_n))d^\alpha(\mu_n),$$
$$(\mu_1) = (x_1(y_1(z_1))), (\mu_i) = (x_i(y_i(z_i))), (\mu_n) = (x_n(y_n(z_n))).$$

Proof. The degree of vertices of PFG is already defined in Lemma 2.7. Applying these definitions, the PFG of the first general Zagreb index for edge rho_1 is:

$$M_\alpha(PFG) = \sum_{i=1}^{n} P(\mu_i)d^\alpha(\mu_i),$$

when $i = 1$,

$$P(\mu_1)d^\alpha(\mu_1) = (P(x_1(y_1(z_1))))d^\alpha(x_1(y_1(z_1))),$$

where $d(\mu_1) = P(\rho_1)$.

When $i = 2$,

$$P(\mu_2)d^\alpha(\mu_2) = (P(x_2(y_2(z_2))))d^\alpha(x_2(y_2(z_2))),$$

where $d(\mu^2) = P(\rho_1) + P(\rho_2)$.

When $i = 3$,

$$P(\mu_3)d^\alpha(\mu_3) = (P(x_3(y_3(z_3))))d^\alpha(x_3(y_3(z_3))),$$

where $d(\mu^3) = P(\rho_2) + P(\rho_3)$.

When $i = n-1$,

$$P(\mu_{n-1})d^\alpha(\mu_{n-1}) = (P(x_{n-1}(y_{n-1}(z_{n-1}))))d^\alpha(x_{n-1}(y_{n-1}(z_{n-1}))),$$

where $d(\mu^{n-1}) = P(\rho_{n-2}) + P(\rho_{n-1})$.

When $i = n$,

$$P(\mu_n)d^\alpha(\mu_n) = (P(x_n(y_n(z_n))))d^\alpha(x_n(y_n(z_n))),$$

where $d(\mu_{n-1}) = P(\rho_{n-1})$.

Hence, by adding all these results, the generalized first Zagreb index for PFG is:

$$M_\alpha(PFG) = (P(x_1(y_1(z_1))))d^\alpha(x_1(y_1(z_1)))+$$
$$\sum_{i=2}^{n-1}(P(x_i(y_i(z_i))))d^\alpha(x_i(y_i(z_i))) + (P(x_n(y_n(z_n))))d^\alpha(x_n(y_n(z_n))).$$

$$M_\alpha(PFG) = (P(\mu_1))d^\alpha(\mu_1) + \sum_{i=2}^{n-1}(P(\mu_i))d^\alpha(\mu_i) + (P(\mu_n))d^\alpha(\mu_n).$$

□

Theorem 3.4. *Let C_n be a Plithogenic cycle graph with $V = \{\mu_1, \mu_2, \mu_3, \ldots, \mu_n\}$ the set of vertices and $E = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_n\}$ be the set of n edges, such that*

$\mu_1 = (x_1(y_1(z)_1))$ *and* $\rho_1 = (x_1(y_1(z)_1))(x_2(y_2(z)_2))$. *Then the Plithogenic version of the first general Zagreb index is:*

$$M_\alpha(PFG) = \sum_{i=1}^{n}(P(\mu_i))d^\alpha(\mu_i), \tag{6}$$

where

$$(\mu_1) = (x_1(y_1(z_1))), (\mu_i) = (x_i(y_i(z_i))), (\mu_n) = (x_n(y_n(z_n))).$$

Proof. The degree of vertices of PFG is already defined in Lemma 2.9. Applying these definitions, the PFG of the first general Zagreb index for vertex μ_1 is:

$$M_\alpha(PFG) = \sum_{i=1}^{n} P(\mu_i)d^\alpha(\mu_i).$$

When $i = 2$

$$P(\mu_1)d^\alpha(\mu_1) = (P(x_1(y_1(z_1))))d^\alpha(x_1(y_1(z_1))),$$

where $d(\mu_1) = P(\rho_n) + P(\rho_1)$.

When $i = 2$,

$$P(\mu_2)d^\alpha(\mu_2) = (P(x_2(y_2(z_2))))d^\alpha(x_2(y_2(z_2))),$$

where $d(\mu^2) = P(\rho_1) + P(\rho_2)$.

When $i = 3$,

$$P(\mu_3)d^\alpha(\mu_3) = (P(x_3(y_3(z_3))))d^\alpha(x_3(y_3(z_3))),$$

where $d(\mu^3) = P(\rho_2) + P(\rho_3)$.

When $i = n - 1$

$$P(\mu_{n-1})d^\alpha(\mu_{n-1}) = (P(x_{n-1}(y_{n-1}(z_{n-1}))))d^\alpha(x_{n-1}(y_{n-1}(z_{n-1})))$$

where $d(\mu^{n-1}) = P(\rho_{n-2}) + P(\rho_{n-1})$.

When $i = n$

$$P(\mu_n)d^\alpha(\mu_n) = (P(x_n(y_n(z_n))))d^\alpha(x_n(y_n(z_n))),$$

where $d(\mu^2) = P(\rho_{n-1}) + P(\rho_n)$.

Hence, by including all these results, the generalized first Zagreb index for PFG is:

$$M_\alpha(PFG) = \sum_{i=1}^{n}(P(x_i(y_i(z_i))))d^\alpha(x_i(y_i(z_i))).$$

$$M_\alpha(PFG) = \sum_{i=1}^{n}(P(\mu_i))d^\alpha(\mu_i).$$

□

Theorem 3.5. *Let* $K_{m,n}$ *be a Plithogenic complete bipartite graph with* $V = \{v_1, v_2, v_3, \ldots, v_m\}$ *and* $U = \{u_1, u_2, u_3, \ldots, u_n\}$ *the set of vertices and* $E = \{\rho_1, \rho_2, \rho_3, \ldots, \rho_n\}$ *be the set of edges, such that* $u_1 = (x_1(y_1(z)_1)), v_1 = (x'_1(y'_1(z')_1))$ *and* $\rho_1 = (x_1(y_1(z)_1))(x'_1(y'_1(z')_1))$. *Then, the Plithogenic version of the first general Zagreb index is:*

$$M_\alpha(PFG) = \sum_{i=1}^{m}(P(v_i))d^\alpha(v_i) + \sum_{j=1}^{n}(P(u_j))d^\alpha(u_j), \tag{7}$$

where

$$(u_n) = (x_n(y_n(z_n))), (v_m) = (x_m(y_m(z_m))).$$

Proof. The degree of vertices of PFG is already defined in Lemma 2.14. Applying these definitions, the PFG of the first general Zagreb index for vertex u_1 is:

$$M_\alpha(PFG) = \sum_{i=1}^{n} P(v_i)d^\alpha(v_i).$$

For vertex set $V = \{v_1, v_2, v_3, \dots, v_m\}$

When $i = 1$,

$$P(v_1)d^\alpha(v_1) = (P(x_1(y_1(z_1))))d^\alpha(x_1(y_1(z_1))),$$

where $d(v_1) = P(v_1u_j)$.

When $i = 2$,

$$P(v_2)d^\alpha(v_2) = (P(x_2(y_2(z_2))))d^\alpha(x_2(y_2(z_2))),$$

where $d(v_2) = P(v_2u_j)$.

When $i = 3$,

$$P(v_3)d^\alpha(v_3) = (P(x_3(y_3(z_3))))d^\alpha(x_3(y_3(z_3))),$$

where $d(v_3) = P(v_3u_j)$.

When $i = m - 1$,

$$P(v_{m-1})d^\alpha(v_{m-1}) = (P(x_{m-1}(y_{m-1}(z_{m-1}))))d^\alpha(x_{m-1}(y_{m-1}(z_{m-1}))),$$

where $d(v_{m-1}) = P(m_{n-1}u_j)$.

When $i = m$,

$$P(v_m)d^\alpha(v_m) = (P(x_m(y_m(z_m))))d^\alpha(x_m(y_m(z_m))),$$

where $d(v_m) = P(v_mu_j)$.

Now, for vertex set $U = \{u_1, u_2, u_3, \dots, u_n\}$,

When $j = 1$,

$$P(u_1)d^\alpha(u_1) = (P(x'_1(y'_1(z'_1))))d^\alpha(x'_1(y'_1(z'_1))),$$

where $d(u_1) = P(u_1v_i)$.

When $j = 2$,

$$P(u_2)d^\alpha(u_2) = (P(x'_2(y'_2(z'_2))))d^\alpha(x'_2(y'_2(z'_2))),$$

where $d(u_2) = P(u_2v_i)$.

When $j = 3$,

$$P(u_3)d^\alpha(u_3) = (P(x'_3(y'_3(z'_3))))d^\alpha(x'_3(y'_3(z'_3))),$$

where $d(v_u) = P(u_3v_i)$.

When $j = n - 1$,

$$P(u_{n-1})d^\alpha(u_{n-1}) = (P(x'_{n-1}(y'_{n-1}(z'_{n-1}))))d^\alpha(x'_{n-1}(y'_{n-1}(z'_{n-1}))),$$

where $d(u_{n-1}) = P(u_{n-1}v_i)$.

When $j = n$,

$$P(u_n)d^\alpha(u_n) = (P(x'_n(y'_n(z'_n))))d^\alpha(x'_n(y'_n(z'_n))),$$

where $d(u_n) = P(u_nv_i)$.

Hence, by adding all these results generalized first Zagreb index for the Plithogenic complete bipartite graph is:

$$M_\alpha(PFG) = \sum_{i=1}^{n}(P(x_i(y_i(z_i))))d^\alpha(x_i(y_i(z_i))),$$

$$M_\alpha(PFG) = \sum_{i=1}^{m}(P(v_i))d^\alpha(v_i) + \sum_{j=1}^{n}(P(u_j))d^\alpha(u_j).$$

□

4. **Applications.**

In this section, an application of the generalized first Zagreb index is discussed. A network of patients of particular lung pleural effusion is taken into account to compute the common symptoms of that disease, where $V(G)$ shows patients of lung pleural effusion with some particular symptoms aggregated by three doctors, $E(G)$ shows the common ratio of symptoms between patients, $d(v_i)$ shows the total ratio of common symptoms. With the help of this application, adding more patients or a common ratio between the patients helps us to compute the common symptoms of lung pleural effusion.

4.1. **Framework of lung pleural effusion.** Consider a Plithogenic network of patients of lung pleural effusion with vertex set as patients affected with pleural effusion where, symptoms of pleural effusion in p_1 are {chest pain, fever, hiccupus}, in p_2 are {Hicuupus, rapid breathing, shortness of breathing}, inp_3 are {chest pain, cough, cold}, and in p_4 are {fever, cough, shortness of breathing}, edge set common symptoms of patients and degree set as common ratio of similar symptoms between the patients. Compute generalized first Zagreb index for Plithogenic framework. The corresponding network is displayed in Figure 6.

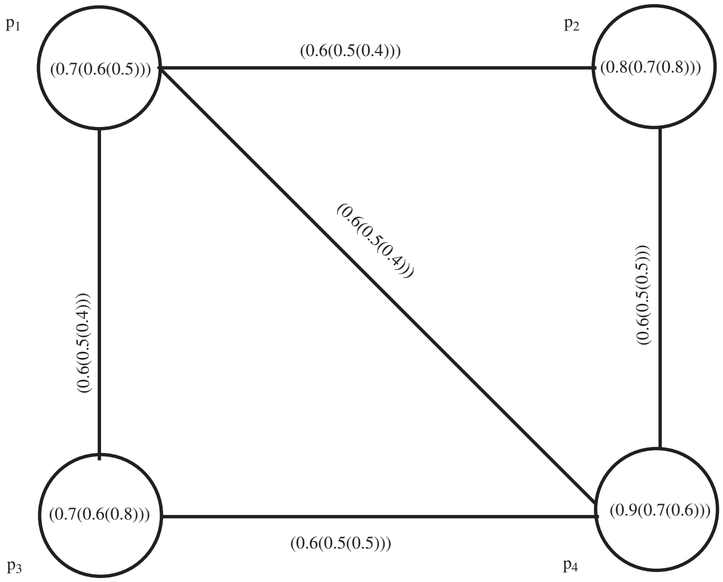

Figure 6. Plithogenic network of the patients for lung pleural effusion.

First, compute the degree of every patient

$$\begin{aligned}
d(v_1) &= (0.6(0.5(0.4))) + (0.6(0.5(0.4))) + (0.6(0.5(0.4))),\\
d(v_2) &= (0.6(0.5(0.4))) + (0.6(0.5(0.5))),\\
d(v_3) &= (0.6(0.5(0.5))) + (0.6(0.5(0.5))) + (0.6(0.5(0.4))),\\
d(v_4) &= (0.6(0.5(0.4))) + (0.6(0.5(0.5))).
\end{aligned}$$

$$\begin{aligned}
d(v_1) &= (1.8(1.5(1.2))),\\
d(v_2) &= (1.2(1.0(0.8))),\\
d(v_3) &= (1.8(1.5(1.4))),\\
d(v_4) &= (1.2(1.0(0.9))).
\end{aligned}$$

Now, compute the 4th power of degree of every vertex

$$\begin{aligned}
d^4(v_1) &= (10.4976(5.0625(0.6561))),\\
d^4(v_2) &= (2.0736(1.0(0.4096))),\\
d^4(v_3) &= (10.4976(5.0625(3.8416))),\\
d^4(v_4) &= (2.0736(1.0(0.6561))).
\end{aligned}$$

4.1.1. *First general Zagreb index.*

$$M_\alpha(PFG) = \sum_{i=1}^{n}(P(x_i(y_i(z_i))))d^\alpha(x_i(y_i(z_i)))$$

$$\begin{aligned}
M_\alpha(PFG) =&(0.7(0.6(0.5)))(10.4976(5.0625(0.6561)))\\
&+ (0.8(0.7(0.8)))(2.0736(1.0(0.4096)))\\
&+ (0.7(0.6(0.8)))(10.4976(5.0625(3.8416)))\\
&+ (0.9(0.7(0.6)))(2.0736(1.0(0.6561))).
\end{aligned}$$

By using basic dot product operation

$$\begin{aligned}
&= 7.34832 + 3.0375 + 0.32805 + 1.65888 + 0.7 + 0.32768\\
&\quad + 7.34832 + 3.0375 + 3.07328 + 1.86624 + 0.7 + 0.39366
\end{aligned}$$

$$= 29.819.$$

4.2. **Framework of lung pleural effusion after addition.** Consider a Plithogenic network of patients with lung pleural effusion with vertex set as patients affected with pleural effusion, after adding another patient or finding more similar symptoms in patients increase the degree of that patient which helps us to find the symptoms can be in the newly affected patient. We need to add an edge and compute the first general Zagreb index of the new graph.

Now, the Plithogenic network of the patient for lung pleural effusion after addition is represented in Figure 7.

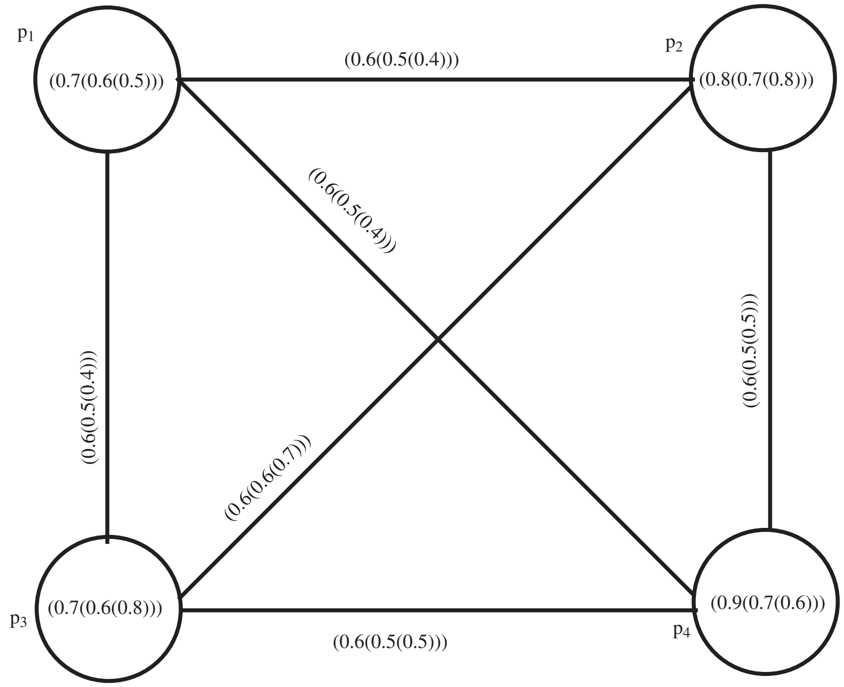

Figure 7. Plithogenic network of the patient for lung pleural effusion after addition.

First, compute the degree of every patient

$$\begin{aligned}
d(v_1) &= (0.6(0.5(0.4))) + (0.6(0.5(0.4))) + (0.6(0.5(0.4))),\\
d(v_2) &= (0.6(0.5(0.4))) + (0.6(0.5(0.5))) + (0.6(0.6(0.7))),\\
d(v_3) &= (0.6(0.5(0.5))) + (0.6(0.5(0.5))) + (0.6(0.5(0.4))),\\
d(v_4) &= (0.6(0.5(0.4))) + (0.6(0.5(0.5))) + (0.6(0.6(0.7))).\\
d(v_1) &= (1.8(1.5(1.2))),\\
d(v_2) &= (1.8(1.6(1.5))),\\
d(v_3) &= (1.8(1.5(1.4))),\\
d(v_4) &= (1.8(1.6(1.6))).
\end{aligned}$$

Now, compute the 4th power of degree of every vertex

$$\begin{aligned}
d^4(v_1) &= (10.4976(5.0625(0.6561))),\\
d^4(v_2) &= (10.4976(6.5536(5.0625))),\\
d^4(v_3) &= (10.4976(5.0625(3.8416))),\\
d^4(v_4) &= (10.4976(6.5536(6.5536))).
\end{aligned}$$

Now, the first general Zagreb index is calculated as follows:

$$M_\alpha(PFG) = \sum_{i=1}^{n}(P(x_i(y_i(z_i))))d^\alpha(x_i(y_i(z_i))),$$

$$\begin{aligned} M_\alpha(PFG) =& (0.7(0.6(0.5)))(10.4976(5.0625(0.6561))) \\ &+ (0.8(0.7(0.8)))(10.4976(6.5536(5.0625))) \\ &+ (0.7(0.6(0.8)))(10.4976(5.0625(3.8416))) \\ &+ (0.9(0.7(0.6)))(10.4976(6.5536(6.5536))). \end{aligned}$$

By using basic dot product operation

$$\begin{aligned} &= 7.34832 + 3.0375 + 0.32805 + 8.39808 + 6.4411 + 4.05 \\ &\quad + 7.34832 + 3.0375 + 3.07328 + 9.4478 + 4.5875 + 3.9321 \\ &= 61.02. \end{aligned}$$

5. **Comparative studies.** In this section, we compare our proposed methodologies with already existing models, including fuzzy [19], intuitionistic fuzzy, picture, and neutrosophic fuzzy graphs. The PFG gives us more generalized results in comparison to all previous environments because of its independent nature. If we try to discuss the same framework of generalized first Zagreb index in an environment of the fuzzy graph then we realize that it has only one aggregated value known as membership value. In the case of an intuitionistic fuzzy graph, we have two as membership and non-membership, while in the case of picture fuzzy graph, we have three aggregated values known as membership, abstinence, and non-membership value but all three values depend on each other because of its restriction $0 \leq \{\text{membership} + \text{abstinence} + \text{non-membership}\} \leq 1$. Furthermore, all of these three environment values are aggregated by one expert but in case PFG there are three or more than three experts who aggregate any scenario. So, because of its independent nature and more number of experts aggregating the attributes we get more generalized results as compared to other environments.

PFG: distinguished itself in the field of graph theory by introducing the PFG and its related properties in a novel way. The PFG offers a more expressive and nuanced representation of interactions in complex systems, encapsulating a wider range of uncertainty than regular fuzzy and intuitionistic fuzzy graphs. Plithogenic sets are added to graph theory to both broaden the theoretical framework and improve modelling capabilities. This makes graph theory a more flexible tool for capturing the inherent ambiguity and vagueness found in real-world situations.

Moreover, the PFG presents a unique method by utilising Plithogenic sets, which intrinsically have a three-layered representation of falsity-membership, indeterminacy, and truth-membership, as opposed to image fuzzy graphs. This subtle addition makes it possible to define uncertainty more precisely, which permits more accurate modelling of complex relationships inside networks.

This work is noteworthy for its comprehensive approach, which includes the introduction of topological indices specifically designed for the Plithogenic fuzzy environment in addition to a novel graph structure. This methodological extension sets it apart from other environments by enabling a thorough investigation of PFGs. The article's applications highlight the usefulness of PFGs in practical settings, demonstrating their adaptability and suitability across a range of fields.

Essentially, the comparison investigations emphasise the unique contributions of the fuzzy graph environment that is Plithogenic, thereby establishing it as a useful component of the fuzzy, intuitionistic, and picture fuzzy graph theories. This paper is a trailblazer in connecting the theoretical and practical elements of PFGs, providing a strong basis for further investigations and applications in decision-making and uncertainty modelling.

6. Conclusion.

To sum up, this innovative work breaks beyond current paradigms by presenting the generic first Zagreb index in the dynamic context of PFGs, a new and wide-ranging network that goes beyond conventional fuzzy and neutrosophic settings. Network structures can be better understood through the conception of PFGs, which include the vertex set, edge set, order, size, and degree set. Specifically, our concepts of complete, complete bipartite, routes, and cycles in the context of Plithogenic environments offer a full toolbox for complex graph analysis. The first general Zagreb index used for individuals suffering from lung pleural effusion highlights the significant practical implications of our approach. The conclusion that more patients or more links improve the capacity to recognise common symptoms highlights the practical significance of our research. This has revolutionary implications for patient care and medical diagnostics in addition to furthering the field of graph theory. Finally, our study goes beyond the theoretical domain, providing a groundbreaking paradigm that both improves the field of graph theory and shows its concrete utility in tackling important medical issues. The path of our research invites more investigation and application in various fields as we move across the nexus of practical healthcare and mathematical abstraction.

7. Future study and open problems.

In this work, we defined the first general Zagreb index for PFG, the researchers can extend it in the following ways:
- Second general Zagreb index for the framework of PFG,
- Sombor indices in the framework of PFG,
- Many existing indices of a crisp graph can be defined in the framework of PFG.

In this article, we introduced the first general Zagreb index for PFG, the following results can also computed under the PFG environment:
- Bounds of first general Zagreb index for PFG,
- Edge and vertex deletion operation of a graph can also be computed for the framework of PFG,
- Edge and vertex addition operation of a graph can also be computed for the framework of PFG.

All in all, other uncertainty-handling approaches such as robust optimization [17, 16] and stochastic optimal control [45, 32] can be also studied for the possible hybridization with the proposed PFG.

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